Hamiltonian systems with dissipation

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1 Lagrangian and Hamiltonian mechanics
   - Lagrangian mechanics
   - Hamiltonian mechanics

2 Introducing dissipation
   - Hamiltonian systems with dissipation
   - Related work
   - The symplectic BEN
   - First consequences

3 A brittle damage model
   - Choosing the Hamiltonian
   - Choosing the dissipation potential
   - The brittle damage model

4 Perspectives
Lagrangian mechanics

- Lagrangian:
  \[ L(t, q, \dot{q}) = \hat{T}(\dot{q}) - \mathcal{E}(t, q) \]

- Evolution equation:
  \[ D_qL(t, q, \dot{q}) - \frac{d}{dt}D_qL(t, q, \dot{q}) = 0 \]
Lagrangian mechanics

- Lagrangian, example:

\[ L(t, q, \dot{q}) = m \frac{||\dot{q}||^2}{2} - (E(q) - f(t)q) \]

- Evolution equation:

\[ -D_q E(q) + f(t) - \frac{d}{dt} (m\dot{q}) = 0 \]
Hamiltonian mechanics

- Hamiltonian:
  \[ H(t, q, p) = T(p) + \mathcal{E}(t, q) \]

- Evolution equation:
  \[
  \begin{align*}
  -\dot{p} & = D_q H(t, q, p) \\
  \dot{q} & = D_p H(t, q, p)
  \end{align*}
  \]
Hamiltonian mechanics

- Hamiltonian, example:

\[ H(t, q, p) = \frac{1}{2m} \|p\|^2 + (E(q) - f(t)q) \]

- Evolution equation:

\[
\begin{align*}
\dot{p} &= - D_q E(q) - f(t) \\
\dot{q} &= \frac{1}{m} p
\end{align*}
\]
Hamiltonian mechanics

- Evolution equation:

\[
\begin{align*}
-\dot{p} &= D_q H(t, q, p) \\
\dot{q} &= D_p H(t, q, p)
\end{align*}
\] (1)

- Let \( J(q, p) = (-p, q) \), then (1) is equivalent with:

\[
\frac{d}{dt}(q, p) + JD_{(q,p)}H(t, q, p) = 0
\]
Let \( z = (q, p) \). Define the Hamiltonian vector field associated to \( H \) as

\[
X H(t, z) = -J D_z H(t, z)
\]

Then (1) is equivalent with:

\[
\dot{z} - X H(t, z) = 0
\]  

Interesting fact: the evolution is conservative, i.e. non-dissipative.

\[
D_z H(t, z) \dot{z} = 0
\]
Two functions: the Hamiltonian $H = H(t, z)$ and a convex dissipation potential $\phi = \phi(z, \dot{z})$.

decompose the evolution into conservative and dissipative parts:

$$\dot{z}(t) = \dot{z}_C(t) + \dot{z}_D(t), \quad \dot{z}_D = \dot{z} - XH(t, z)$$ (3)

Evolution equation:

$$\dot{z}_D \in \partial^\omega (\phi(z, \cdot)) (\dot{z})$$ (4)
Evolution equation:

\[ \dot{z}_D \in \partial^\omega (\phi(z, \cdot))(\dot{z}) \]

Comments:

- Introduced in M. Buliga, Hamiltonian inclusions with convex dissipation with a view towards applications, Mathematics and its Applications 1, 2 (2009), 228-251, arXiv:0810.1429
- The notation \( \partial^\omega \phi \) means the symplectic subgradient of \( \phi \).
In the Lagrangian formalism this can be traced back to Rayleigh and Kelvin


autonomous Hamiltonian systems with a Rayleigh dissipation function added:

Related

- other proposals for generalizations of lagrangian or hamiltonian mechanics by multivariate analysis:
- may be related to Aubin’s viability theory, but not clear how, especially in the case of 1-homogeneous $\phi$.
- it looks like a dynamic version of Mielke theory of quasistatic rate-independent processes, in this case of 1-homogeneous $\phi$
Introducing dissipation

- Evolution equation:
  \[ \dot{z}_D \in \partial \omega \left( \phi(z, \cdot) \right)(\dot{z}) \]

- Work in progress with Géry de Saxcé on a more elegant approach called "the symplectic Brezis-Ekeland-Nayroles principle",


- Based on the symplectic Fenchel inequality
  \[ \psi(z) + \psi^*(\omega(z')) \geq \omega(z, z') \]
Consequences

- Evolution equation:

\[
\dot{z}_D \in \partial^\omega (\phi(z, \cdot))(\dot{z})
\]

- the evolution equation is equivalent with

\[
J \dot{z} - D_z H(t, z) \in \partial (\phi(z, \cdot))(\dot{z})
\]

i.e. for any \( u \)

\[
\phi(z, \dot{z} + u) \geq \phi(z, \dot{z}) + \langle J\dot{z} - D_z H, u \rangle
\]

- suppose that \( \phi(z, 0) = 0 \) and \( \phi(z, \dot{z}) \geq 0 \), then take \( u = -\dot{z} \)

\[
0 = \phi(z, 0) \geq \phi(z, \dot{z}) + \langle J\dot{z} - D_z H, -\dot{z} \rangle = D_z H(t, z)\dot{z}
\]

therefore the system dissipates!

\[
D_z H(t, z)\dot{z} \leq 0
\]
Particular cases

- Evolution equation:
  \[ \dot{z}_D \in \partial \omega (\phi(z, \cdot)) (\dot{z}) \]

- \( \phi = 0 \), of course this is Hamiltonian mechanics.

- \( \phi(z, \dot{z}) = \| \dot{z} \|^2 \), this is Rayleigh dissipation

- \( \phi(z, \dot{z}) = \| \dot{z} \| \), this gives a dynamic version of Mielke theory of quasistatic rate-independent processes

- \( H = 0 \) gives an interesting fixed point problem:
  \[ \dot{z} \in \partial \omega (\phi(z, \cdot)) (\dot{z}) \]
Variational approximation of a Mumford-Shah energy


\[
E_c(u, d) = \int_{\Omega} \left\{ \phi(d) w(\nabla u) + \frac{1}{2} \gamma c |\nabla d|^2 + \frac{\gamma}{2c} d^2 \right\} \tag{5}
\]

is a variational approximation, as \(c \to 0\) of the Mumford-Shah energy:

\[
E(u, S) = \int_{\Omega} w(\nabla u) \, dx + \gamma \mathcal{H}^2(S) \tag{6}
\]
On the Mumford-Shah Energy

Starting with the foundational papers of Mumford, Shah / De Giorgi, Ambrosio / Ambrosio / the development of models of quasistatic brittle fracture based on Mumford-Shah functionals continues with Francfort, Marigo / Mielke / Dal Maso, Francfort, Toader / Buliga.

All these models are based on a technique of time discretization followed by a sequence of incremental minimization problems. These models are either seen as applications

- of De Giorgi method of energy minimizing movements,
- or in the frame of the theory of Mielke of rate-independent evolutionary processes.

\[
E(u, S) = \int_\Omega w(\nabla u) \, dx + \gamma \mathcal{H}^2(S)
\]
A brittle damage model

Choosing the Hamiltonian

A good energy for damage

- Let’s take seriously this functional as a good energy for a brittle damage model:

\[ E_c(u, d) = \int_\Omega \left\{ \phi(d) w(\nabla u) + \frac{1}{2} \gamma c |\nabla d|^2 + \frac{\gamma}{2c} d^2 \right\} \]  

(7)

because \( d \in [0, 1] \) is good for a damage variable.

- here \( \phi \) is a decreasing function from \([0, 1]\) to \([0, 1]\) such that \( \phi(0) = 1 \) and \( \phi(1) = 0 \)

- \( \phi(d) w(\nabla u) \) is the damaged elastic energy density

- \( \frac{1}{2} \gamma c |\nabla d|^2 + \frac{\gamma}{2c} d^2 \) is a nonlocal damage energy density, compatible with

The Hamiltonian

- $q = (u, d)$ and $p = (p, y)$
- Let us define the the Hamiltonian as:

$$H(t, q, p) = E_c(u, d) + T(p, y) - \langle l(t), u \rangle \quad (8)$$

- where the kinetic energy is

$$T(p, y) = \int_{\Omega} \left[ \frac{1}{2} \gamma c |y|^2 + \frac{1}{2\rho} \|p\|^2 \right]$$

- $\gamma c$ is a microinertia scalar, cf. Stumpf and Hackl
The dissipation potential

- the dissipation potential is inspired from

\[ \phi = \phi(\dot{d}) = \int_{\Omega} \left[ \chi_{[0,1]}(d) + \chi_{[0,\infty)}(\dot{d}) + \beta \left| \dot{d} \right| \right] \]  

- it depends only on the ”dissipative variable” $\dot{d}$. 
The equations of the model

- The equations coming from the "non-dissipative variables" \((u, p)\) are the usual balance equations and boundary conditions, like

\[
div \left( \phi(d) Dw(\nabla u) \right) + f(t) = \dot{p}
\]

- Because \(\phi = \phi(\dot{d})\) we get

\[
p = \rho \dot{u}
\]

\[
\dot{d} = \gamma cy
\]

and ...
The equations of the model

... and for all \( \hat{d} \), such that \( \hat{d}(x) + \dot{d} \geq 0 \)

\[
\beta \int_{\Omega} \left[ | \dot{d} + \hat{d} | - | \dot{d} | \right] \geq \frac{1}{c} \gamma \int_{\Omega} \left[ (\frac{\gamma}{c} d + \phi'(d)w(\nabla u) + \dot{y}) \hat{d} + \gamma c \nabla d \cdot \nabla \hat{d} \right].
\]
Eventually we get the following constitutive law of brittle damage evolution:

\[- (\ddot{d} + \gamma^2 d + \gamma c \phi'(d) w(\nabla u) - \gamma^2 c^2 \Delta d) \in \begin{cases} 
\gamma c \beta \\
(-\infty, \gamma c \beta] 
\end{cases} , \quad \dot{d} > 0, \quad \dot{d} = 0\]
What else?

The same can be done (work in progress with Géry de Saxcé) for:

- plasticity
- friction
- ... you name it and we may try to do it!