1 Purpose of the visit

The research project described further could be done in very few places, the IHES being in my opinion the most fitted (see Motivation section below).

The new field of non-commutative (or non-euclidean) analysis could emerge in the next years, if properly supported. The name "non-euclidean" is given to suggest a parallel to non-euclidean geometry, which proved eventually useful in physics. The line of research which is proposed here could turn out to be trivial, but it could also say something interesting about the world we are part of. The complete gratuity of the subject is a justification to classify it as fundamental research.

There are several ways to associate a tangent bundle to a metric measure space (Cheeger [7] for what seems to be a cotangent bundle). For a sub-Riemannian manifold, after the deep construction in Bellaïche [1], last section, we find other variants of a tangent bundle (Margulis & Mostow [11], [12], Vodop’yanov, Greshnov [17]) or Buliga [4], [2]. These bundles differs and the constructions are of a nontrivial level of sofistication. Analysis on his type of spaces is based on these bundles.

Pansu [13] type Rademacher theorem implies that any open set of a Carnot group endowed with a left invariant Carnot-Carathéodory distance cannot be mapped by a locally Lipschitz function to an euclidean space. This means: if a "chart" is a quasi-isometric function then there is a limited precision for any euclidean chart of any open part of a Heisenberg group. The notion of "precision" is related to Gromov-Hausdorff distance.

A physicist mind, after learning about such results, could ask ”why we suppose that the configuration space, or the phase space, or the space-time admit euclidean charts with arbitrary precision?” Any of these spaces could be in fact of non-euclidean type, ”at any scale” in the sense that they admit a natural dilatation structure such that there is no (a.e.) differentiable function from these spaces to an euclidean space. They could have a non-euclidean analysis.

In cases where contradictions appear if we suppose that (for example) the phase space is euclidean at any scale, non-commutative geometry shows ways to look instead to non-commutative versions of objects related to the space.

We propose instead a frontal approach: look at the space, it exists, only that it has an analysis which is non-commutative.
2 Motivation

Apparently most (and there are not many) researchers working in sub-Riemannian geometry, after a time they change their job. In the same time sub-Riemannian geometry regularly appears in various disguises, again and again. This is a clear sign that the subject is ahead of time.

As a researcher in sub-Riemannian geometry, I propose that IHES, with its well-known open spirit, supports the "Non-Euclidean Analysis" start-up.

My current collaborations on this subject are with:

- Martin Reimann (martin.reimann@math-stat.unibe.ch) and his team at the Mathematical Institute, University of Bern. In the Borel Seminar 2003, organized by them at Bern, sub-Riemannian geometry had a central part.

- Sergey Vodop’yanov (vodopis@math.nsc.ru) from Sobolev Institute, Novosibirsk. We are writing together the paper in preparation [6].

This project has been written at IHES, during a very short visit. I want to express my thanks for this. The purpose of the visit was to meet Gromov, which unfortunately was not possible. Nevertheless, I had the opportunity to contact Andre Bella¨ıche and Pierre Pansu. Bella¨ıche showed interest in future collaboration. In my opinion his paper [1] is one of the best in the field.

3 Example and background

In this section we start with an example in sub-Riemannian geometry. The next subsection is dedicated to the background of the research proposal.

3.1 Example: Hamiltonian dynamical systems

We denote by $H(n) = \mathbb{R}^{2n} \times \mathbb{R}$ the Heisenberg group and by $W(n) = \mathbb{R}^{2n} \times S^1$ the Weyl group. The operations on these groups are:

$$\forall (x, x'), (y, y') \in H(n) \quad (x, x')(y, y') = (x + y, x' + y' + \frac{1}{2}\omega(x, y))$$

$$\forall (x, e^{ix'}), (y, e^{iy'}) \in W(n) \quad (x, e^{ix'})(y, e^{iy'}) = (x + y, e^{i[x' + y' + \frac{1}{2}\omega(x,y)]})$$

Here $\omega$ is the canonical symplectic form on $\mathbb{R}^{2n}$.

Dilatation structure. For any sufficiently small $\varepsilon > 0$ define the dilatations:

$$\forall \tilde{x} = (x, x') \in H(n) \quad \delta_\varepsilon \tilde{x} = (\varepsilon x, \varepsilon^2 x')$$

$$\forall \tilde{x} = (x, e^{ix'}) \in U_\varepsilon \subset W(n) \quad \delta_\varepsilon \tilde{x} = (\varepsilon x, e^{i\varepsilon^2 x'})$$

Here $U_\varepsilon$ is a neighbourhood of the neutral element in $W(n)$, such that the dilatation has a well defined inverse. Dilatations are group morphisms, where the definition makes sense.
Take by comparison the commutative group \((\mathbb{R}^m, +)\). Here dilatations are the usual
\[
\delta_\varepsilon(x) = \varepsilon x
\]

**Carnot-Carathéodory distance.** We take \(D\) to be a bracket generating vector space in the Lie algebra of the group. For the Heisenberg group the exponential map is the identity, therefore we identify the Lie algebra with the group, as sets. We take then
\[
D = \mathbb{R}^{2n} \times \{0\}
\]
For the Weyl group the exponential is the map
\[
(X, X') \mapsto (X, e^{iX'})
\]
We take the same generator set \(D\) as previously.

For the abelian \(\mathbb{R}^m\) again the exponential is the identity. This time we take \(D\) equal to \(\mathbb{R}^m\).

Let us denote any of the groups \((\mathbb{R}^m, +), H(n), W(n)\) by \(N\). The group operation will be denoted by multiplication. (Therefore if \(x, y \in \mathbb{R}^m\) then we write \(xy\) for \(x + y\) and \(xy^{-1}\) for \(x - y\). In this strange notation \(x0 = x\).) The neutral element of \(N\) is denoted by \(0\).

On the generating set \(D\) we put an euclidean norm \(\|\cdot\|\). We transport by (differential of) left translations
\[
L_x(y) = xy
\]
the generating set and the euclidean norm all over the group, obtaining therefore a sub-bundle of the tangent bundle of the group seen as a manifold. We call this sub-bundle a horizontal distribution and we denote it with the same letter \(D\).

A curve in the group \(N\) is called horizontal if it is almost everywhere tangent to the distribution \(D\).

We define then, for any \(h > 0\), the \(h\) Carnot-Carathéodory distance in the group \(N\) to be the left invariant distance
\[
d^h(x, y) = d^h(0, x^{-1}y)
\]
given by the expression
\[
d^h(0, x) = \inf \left\{ \int_0^1 h \|\dot{c}(t)\|_{c(t)} \, dt \right\}
\]
where the infimum is taken over all horizontal curves which join \(0\) and \(x\).

We shall suppose that the exponential map is injective on the closed unit ball in \(D \subset Lie N\), with center 0, with respect to the norm \(\|\cdot\|\). We shall denote the exponential of this closed ball by \(B\). The set \(B \subset N\) generates the group \(N\).

The dilatations are compatible with the distance \(d^h\) in the sense that for small \(\varepsilon\) and \(x, y \in N\) close enough to 0 we have
\[
d(\delta_\varepsilon x, \delta_\varepsilon y) = d(x, y)
\]
**Word tangent bundle.** (Buliga, sections 3.4 and 4, [4]) A word in $N$ is denoted by:

$$w = \left( \begin{array}{cccc} x_1 & x_2 & \cdots & x_p \\ y_1 & y_2 & \cdots & y_p \end{array} \right)$$

where $x_i, y_i \in B$ for all $i = 1, \ldots, p$. To the word $w$ is associated the function (denoted also by $w$) defined for sufficiently small $\varepsilon > 0$ by

$$w(\varepsilon) = x_1 \delta_\varepsilon(y_1) x_2 \delta_\varepsilon(y_2) \cdots x_p \delta_\varepsilon(y_p)$$

Two words with the same associated function will be identified. To the empty word is associated the constant function which maps $\varepsilon$ to the neutral element of $N$.

Words can be concatenated (which correspond to multiplication of the associated functions). The empty word is a neutral element for this operation. The inverse of the word $w$ is the word with associated function

$$w^{-1}(\varepsilon) = (w(\varepsilon))^{-1}$$

We shall further identify the words $w_1$ and $w_2$ if

$$\delta_\varepsilon^{-1} \left( w_1^{-1}(\varepsilon) w_2(\varepsilon) \right) \to 0$$

as $\varepsilon$ converges to 0.

The concatenation of words operation and the inverse resist after the identification and we obtain a group of words, denoted by $WN$ and called the word tangent bundle of $N$. This is a bundle over $N$ by the map

$$w \in WN \rightarrow w(0) \in N$$

We shall denote by $W_0N$ the fiber over 0, that is the class of words $w$ such that $w(0) = 0$. The elements of $W_0N$ are called word vectors. They form a group.

In the case $N = \mathbb{R}^m$ we obtain the trivial tangent bundle to $\mathbb{R}^m$. An element of this bundle is a function

$$w(\varepsilon) = x + \varepsilon y$$

which corresponds to a 2 letter word. Word vectors are nothing but vectors, in the sense that any word vector has the form $\varepsilon y$ with $y \in \mathbb{R}^m$.

In the cases $N = H(n)$ or $N = W(n)$ we have a richer word tangent bundle. Any element of the word tangent bundle can be written as a 4 letter word. Any word vector can be represented as a 3 letter word. More specifically, any word vector has the associated function

$$w(\varepsilon) = \exp(\varepsilon x, \varepsilon x' + \varepsilon^2 x'')$$

The dimension of the group of word vectors equals the Hausdorff dimension of the metric space $(N, d^n)$.

In all cases considered the word vectors group $W_0N$ can be identified with a finite dimensional space of polynomials in $\varepsilon$. We shall use further the identification between word vectors and their polynomial representatives. The words "uniform" or "compact" are considered with respect to the coefficients of the polynomials.
Word derivatives. (Buliga def. 4.6, [4]) Take a function \( f : N_1 \to N_2 \). We say that \( f \) is derivable in the point \( x \) in \( N_1 \) if there is a group morphism

\[
Df(x) : W_0N_1 \to W_0N_2
\]

and a small enough compact neighbourhood \( K \) of the 0 word vector in \( N_1 \) such that

\[
\frac{1}{\varepsilon}d_2(f(xw(\varepsilon)), f(x)Df(x)(w)(\varepsilon)) \to 0
\]

uniformly with respect to \( w \in K \).

This generalizes the Pansu [13] derivative, which is defined only over one letter word vectors, of the form \( \delta_\varepsilon x \).

No Lipschitz charts. As a metric space \( (N, d^h) \) has a finite Hausdorff dimension \( Q \). For the Heisenberg \( H(n) \) and Weyl \( W(n) \) groups this dimension is \( Q = 2n + 2 \), which is bigger than the topological dimension \( 2n + 1 \). (But in all cases \( Q \) is the dimension of the word vector group).

The Hausdorff measure of dimension \( Q \) is left invariant (like the distance) and right invariant, absolutely continuous with respect to Lebesque measure, hence it is a Haar measure on \( N \).

By "a.e." we mean almost everywhere with respect to this measure.

The function \( f : A \subset N_1 \to N_2 \) is Lipschitz if there is a positive constant \( C \) such that for all \( x, y \in A \) we have

\[
d_2(f(x), f(y)) \leq Cd_1(x, y)
\]

The function \( f \) is bi-Lipschitz if there are positive constants \( C, C' \) such that

\[
C'd_1(x, y) \leq d_2(f(x), f(y)) \leq Cd_1(x, y)
\]

The Pansu-Rademacher theorem is, in the case \( N_1 \) and \( N_2 \) being the Heisenberg or abelian groups, the following.

**Theorem 3.1** (Pansu thm. 2, [13]) Let \( A \subset N_1 \) be open and \( f : A \to N_2 \) Lipschitz. Then \( f \) is Pansu derivable a.e.

The theorem does apply for the Weyl group, remarking that this group is locally isometric with the Heisenberg group.

**Theorem 3.2** If \( N \) is the Heisenberg or Weyl group and \( A \subset N \) is open then there is no bi-Lipschitz function \( f : A \subset N \to \mathbb{R}^m \).

A function \( f : A \subset N_1 \to N_2 \) is a quasi-isometry if there are positive constants \( C, \lambda > 0 \) such that for all \( x, y \in A \) we have

\[
|d_2(f(x), f(y)) - Cd_1(x, y)| \leq \lambda
\]

In contrast with the previous theorem we have
Theorem 3.3 There is a quasi-isometry from the Euclidean $\mathbb{R}^{2n}$ to the Weyl group $(W(n), d^h)$, with constants $C = 1$, $\lambda = \sqrt{2\pi}h$.

Suppose that we have a dynamical system which lives in the Weyl group and we want to make a chart of the Weyl group using a $d^h$ ruler, in order to understand how the dynamical system moves. The theorem says that we can do this chart with a maximum precision proportional with $h$. Does it sound familiar?

Concerning the word derivative, let us introduce a weak version of it: the function $f : N_1 \rightarrow N_2$ is word derivable in the measurable sense if there is a map $Df : N_1 \rightarrow Hom(W_0N_1, W_0N_2)$ which is defined almost everywhere in $N_1$ such that for any function $g : N_1 \rightarrow \mathbb{R}$ which is Lipschitz, with support compact, we have

$$\frac{1}{\varepsilon} \int_{N_1} g(x) d_2(f(xP(\varepsilon), f(x)Df(x)(P)(\varepsilon))\ dx \rightarrow 0$$

uniformly with respect to $P$, when $\varepsilon$ converges to 0. Remark that $Df$ is not uniquely defined, but we can find out more about this by studying the class of Lipschitz functions with compact support from $N_1$ to $\mathbb{R}$.

We have then:

Theorem 3.4 (Buliga, in preparation) Let $f : N_1 \rightarrow N_2$ be a Lipschitz function. According to Pansu Rademacher theorem, the function is almost everywhere Pansu derivable.

Then for almost every $x \in N_1$ the Pansu derivative $Df(x)$ extends to a group morphism $Df(x) \in Hom(W_0N_1, W_0N_2)$ such that $f$ is word derivable in the measurable sense, with the extension $Df$ as a derivative.

To end this paragraph, let us remark that left and right translations in any group $N$ are word derivable (but only left translations are Pansu derivable). Also, the word tangent bundle of a Carnot group is Carnot.

Hamiltonian dynamics. Here by "smooth" we mean word derivable, not Pansu derivable.

Theorem 3.5 (Buliga section 5, [4]) Let $t \mapsto \tilde{\phi}_t : W(n) \rightarrow W(n)$ be a dynamical system with compact support, such that:

- for almost any $t$ the map $\tilde{\phi}_t$ is measure preserving and a.e. smooth,
- for almost all $\tilde{x} = (x, e^{ix'}) \in W(n)$ the trajectory $t \mapsto \tilde{\phi}_t(\tilde{x})$ is a.e. smooth.

Then:

- a.e. trajectory has Hausdorff measure equal to 2.
For a.e. $\tilde{x} = (x, e^{ix'})$ the Hausdorff measure $\mathcal{H}^2$ restricted to the trajectory which passes by $\tilde{x}$ has constant density with respect to $dt$ and this density is a positive function $H(x)$.

Finally, the flow $\tilde{\phi}_t$ admits the representation

$$\tilde{\phi}(x, e^{ix'}) = (\phi_t(x), e^{i[x'+F_t(x)]})$$

where $t \mapsto \phi_t$ is a Hamiltonian flow which locally admits $H$ as a Hamiltonian. $F_t$ is the generating function of $\phi_t$.

There are several applications of this theorem. One of them is to give a short geometrical proof to the fact that the Hofer distance is non-degenerate (Buliga section 5, [4]). Another is to give, by using the last theorem from previous paragraph, an averaged form for any Lipschitz measure preserving function.

**Critical view of the classical approach.** Geometric analysis in its classical setting is missing all this. The reason is simple: if you look only in the horizontal direction then you shall find only euclidean-like properties. I don’t know of any strong use of the fact that there are curves in Carnot groups which have Hausdorff measure bigger than one. In [4] we give as an immediate application the non-degeneracy of the Hofer distance.

Another critic concerns the Lie groups in sub-Riemannian setting. In any of the non-commutative groups $N$ described above, the right translations are not smooth according to Pansu derivative. Therefore in this classical setting the notion of Lie group is void. And the notion of tangent bundle to such a group is difficult.

### 3.2 Background

**Metric profiles and curvature of metric space.** (Buliga [3], [2])

Given $(X, d)$ a locally compact metric space, its metric profile at $x \in X$ is the function

$$\varepsilon \in (0, 1] \mapsto \mathbb{P}_m(x, \varepsilon) = [\hat{B}(x, 1), x, \frac{1}{\varepsilon}d]$$

where $[Y, y, d]$ denotes the isometry class of the pointed metric space $(Y, d)$. The metric profile is a curve in the space $CMS$ of (isometry classes of) compact metric spaces, seen as a metric space with the Gromov-Hausdorff distance (Gromov [9]).

We can define a notion of metric profile regardless to any distance.

**Definition 3.6** A metric profile is a curve $\mathbb{P} : [0, a] \to CMS$ such that:

(a) it is continuous at 0,

(b) for any $b \in [0, a]$ and fixed $\varepsilon \in (0, 1]$ we have

$$d_{GH}(\mathbb{P}(\varepsilon b), \mathbb{P}^m_{d_b}(\varepsilon, x)) = O(b)$$
We used here the notation $P(b) = [\bar{B}(x,1), d_b]$ and $P^m_d(\varepsilon, x) = [\bar{B}(x,1), \frac{1}{\varepsilon}d_b]$.

The metric profile is nice if

$$d_{GH}(P(\varepsilon b), P^m_d(\varepsilon, x)) = O(\varepsilon)$$

If $(X, d)$ admits a tangent space at $x$ then the metric profile can be prolonged by continuity at $\varepsilon = 0$. In this case the curvature of $(X, d)$ at $x$ is by definition the rectifiability class at $\varepsilon = 0$ of the metric profile with respect to the Gromov-Hausdorff distance. This curvature can be classified by (comparison with the metric profile of) homogeneous metric spaces of a privileged type (i.e. previously chosen according to geometer’s tastes).

The space $(X, d)$ might come from a geometric construction, for example $d$ might be the Carnot-Carathéodory distance associated to the sub-Riemannian manifold $(X, D, g)$. In such cases one has several ways to deform the distance $d$ around the point $x \in X$, for example considering rescallings of privileged coordinates around $x$, or of privileged frames. Each such deformation leads to a metric profile function.

All such homogeneous spaces are quotients of groups endowed with a metric structure. This calls for a close examination of such groups and their quotients.

In [?] we give an algorithm for the classification of curvatures of a regular sub-Riemannian manifold. We apply this algorithm for the Riemannian case and for the 3 dimensional contact case, in order to show that we fall over known definitions of curvature.

**Sub-Riemannian Lie groups and tangent bundles.** (Buliga [5], [3])

The extra structure of sub-Riemannian manifolds permitted to Margulis and Mostow [12] to construct a tangent bundle to a sub-Riemannian manifold. Their previous paper [11] contains a study of the differentiability properties of quasi-conformal mappings between regular sub-Riemannian manifolds. The central result is a Stepanov theorem in this setting, that is any quasi-conformal map from a Carnot-Carathéodory manifold to another is a.e. differentiable. Same result for quasi-conformal maps on Carnot groups has been first proved by Koranyi, Reimann [10].

As a motivation of the paper [12] the authors mention a letter from Deligne about the fact that in [11] they refer to ”the tangent space” to a point, that is to a tangent bundle structure, which was not constructed in the first paper. They remedy this gap in the second paper [12]; their tangent bundle has fibers isometric with the tangent spaces.

For (almost) general metric spaces Cheeger [7] constructed a tangent bundle. The tangent bundle constructed by Cheeger does not have as fiber the metric tangent space, in the case of regular sub-Riemannian manifolds.

Vodop’yanov and Greshnov pursued this path and propose in [16], [17] another construction of a tangent bundle of a regular sub-Riemannian manifold.

In Buliga [4] is presented the formalism of uniform groups as an attempt to show that sub-Riemannian geometry is in fact non-Euclidean analysis, from the moment
when one looks up from the horizontal direction, or when one tries to construct non-local objects like the tangent bundle.

Recall that from previous considerations we are interested in groups endowed with a metric structure. A typical situation is a Lie group \( G \) endowed with a left invariant distribution \( D \) constructed from a subspace \( D \subset \text{Lie} \ G \) which bracket generates the whole algebra. By an arbitrary choice of an Euclidean norm on \( D \) we can endow \( G \) with a left invariant Carnot-Caratheodory distance. The machinery of metric spaces comes into action and tells that, as a metric space, \( G \) has tangent space at any point and any such tangent space is isomorphic with the nilpotentization of \( g \) with respect to \( D \), denoted by \( N(G) \).

\( N(G) \) is a Carnot group, that is a simply connected nilpotent group endowed with a one-parameter group of dilatations. The intrinsic calculus on \( N(G) \) is based on the notion of derivative introduced by Pansu [13]. We shall say that a function is Pansu smooth if it is Pansu derivable. The same denomination will be used for functions between Carnot groups.

The following comments can be made.

a) Right translations, the group operation and the inverse map are not Pansu smooth.

b) If \( G \neq N(G) \) then generically there is no atlas of \( G \) over \( N(G) \) with smooth transition of charts such that the group exponential is Pansu smooth.

c) Mostow-Margulis construction of a tangent bundle \( TG \) of \( G \) do not provide a group structure to \( TG \). In contrast with this, in the case of Carnot groups, the word tangent bundle has such a structure. In the case of a general group the notion of mild tangent bundle (Buliga [4]) should be considered.

d) there is no notion of higher order Pansu derivatives (except the horizontal classical derivatives which have no direct intrinsic meaning). In contrast with this, in the word tangent bundle one can differentiate at will. (In fact the second order derivative give interesting informations about the convexity of the function, as usual).

**Dilatation structures.** (Buliga [2], Buliga, Vodop’yanov [6])

A dilatation structure is the central notion of interest in the study of the differentiability properties of a sub-Riemannian manifold.

Suppose that in a neighbourhood of a point \( x \in X \), \((X,d)\) locally compact metric space, we have a one parameter family of functions \( \delta^\varepsilon_x \). Then we can transport the distance \( d \) to a distance denoted by \((\delta,\varepsilon)\).

**Definition 3.7** A dilatation structure associated to \((X,d)\) is an assignment \( x \in X $\mapsto \delta^\varepsilon_x : \mathcal{O}(x) \rightarrow \mathcal{O}(x) \), for all \( \varepsilon \in (0,1) \), where \( \mathcal{O}(x) \) is an open contractible neighbourhood of \( x \) and all \( \delta^\varepsilon_x \) are invertible, such that:

(a) for any \( x \in X \) the map

\[
P^\delta(x)(\varepsilon) = \frac{[\mathcal{B}_d(x,\varepsilon),x,(\delta,\varepsilon)]}{\mathcal{B}_d(x,\varepsilon)}
\]
is a nice metric profile,

(b) if we denote by $\mathbb{P}^m(x)$ the metric profile of the space $(X,d)$ at $x$ then
\[
\lim_{\varepsilon \to 0} d_{GH}(\mathbb{P}^m(x)(\varepsilon),\mathbb{P}^d(x)(\varepsilon)) = 0
\]

(c) for any $x \in X$ and $y \in O(x)$ the map
\[
\left(\delta^{-1}_\varepsilon \circ \delta^{x}_\varepsilon \delta^{y}_\varepsilon \right)
\]
converges uniformly to a map, as $\varepsilon$ tends to $0$. The convergence is uniform with respect to $x$ in a compact set.

(d) for small enough $\alpha > 0$ we have the uniform limit
\[
\delta^{y}_\alpha \delta^{x}_\varepsilon \rightarrow \delta^{x}_\alpha
\]
which is uniform with respect to $x$ and $u \in O(x)$, both in compact sets.

(e) $\delta^x_\varepsilon$ contracts $O(x)$ to $x$, uniformly with respect to $x$ in compact sets.

This definition generalizes the axioms for "uniform groups" introduced in Buliga [4], section 3.1.

The meaning of point (c) in definition 3.7 is that approximate infinitesimal translations converge. This allows to construct a tangent bundle.

**Definition 3.8** The virtual tangent bundle associated to a dilatation structure is the assignment
\[
x \in X \mapsto V_T^\delta_x X = \left\{ \lim_{\varepsilon \to 0} \left(\delta^{-1}_\varepsilon \circ \delta^{x}_\varepsilon \delta^{y}_\varepsilon \right) : y \in O(x) \right\}
\]

Fix $x$ and denote
\[
u \ast v = \lim_{\varepsilon \to 0} \left(\delta^{-1}_\varepsilon \circ \delta^{x}_\varepsilon \delta^{y}_\varepsilon \right)
nu
\]
Then $u \ast v$ is not an operation, but it leads to an operation, if we think at $u \ast v$ as the left translation of $v$ by "$u^{-1}$". Define for every $L \in V_T^\delta_x X$
\[
\delta^{L}_\varepsilon \circ \delta^{-1}_\varepsilon \circ L \circ \delta^{-1}_\varepsilon \circ L
\]

**Theorem 3.9** For any $\varepsilon > 0$ sufficiently small and any $L \in V_T^\delta_x X$ we have $\delta^{L}_\varepsilon \circ \delta^{-1}_\varepsilon$. Moreover $V_T^\delta X$ forms a conical uniform group in the sense of definition 3.3 Buliga [4].

The proof is a transcription of the proof of proposition 3.4 op.cit..

In the case of sub-Riemannian spaces more it is true: consider the dilatation structure induced by a normal frame. Then $V_T^\delta_x M$ is isomorphic as a group with the metric tangent space at $x$ cf. Buliga [4] section 3.2.
Quantum dynamical systems. (Buliga [2])

To say it in few words: a quantum dynamical system is just a dynamical system \( t \mapsto \phi_t : X \to X \) in a metric measure space \((X, d, \mu)\) endowed with a dilatation structure \( \delta \). The dynamical system has to be:

(A) measure preserving,

(B) smooth (orbits and the transformations \( \phi_t \) should be \( \delta \) derivable),

(C) the orbits

\[ \{ \phi_t(x) : t \in [a, b] \} \]

have Hausdorff dimension 2. The Hausdorff measure 2 should be absolutely continuous with respect to \( t \) and the density of this measure is by definition the Hamiltonian.

Look for example to the case of a \( S^1 \)-bundle associated to a pre-quantum contact manifold. We see see this as a contact sub-Riemannian manifold.

If the pre-quantum contact manifold is a pre-quantization of an integral symplectic manifold then any dynamical system which satisfies (A), (B), is a lifting of a Hamiltonian dynamical system on the symplectic manifold. Moreover, in this case the condition (C) is satisfied, in the sense that the orbits of the dynamical system have indeed Hausdorff dimension 2 and the density of the Hausdorff measure 2 with respect to the (transport from \( \mathbb{R} \) of the) Lebesgue measure 1 on the curve is a Hamiltonian for the dynamical system on the symplectic manifold.

A measurement process should correspond to trying to make an Euclidean chart of this dynamical system. If the space is not Euclidean at any scale (as the sub-Riemannian Heisenberg group, for example) then such a map is impossible to be done exactly (i.e. in a derivable way).

Planck constant might be the effect of this fact, namely it could measure the distance from the (metric profile or dilatation profile) and best Euclidean approximations. After reading the paper [2] one can be sensible to the idea that the Planck constant could measure a distance between curvatures.

References


