Non-euclidean analysis of dilation structures

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Imagine that the metric space \((X, d)\) represents a territory. We want to make maps of \((X, d)\) in the metric space \((Y, D)\) (a piece of paper, or a scaled model).
We need many maps, at several scales $\varepsilon_1 > \varepsilon_2 > \ldots > \varepsilon_n$.

An atlas of compatible maps, a manifold?
Exploring space

A model of a map of \((X, d)\) in \((Y, D)\) is a relation \(\rho \subset X \times Y\).

How good is the map? Look at: \(|d(u, v) - D(u', v')|\).
Exploring space

Accuracy: "closeness of agreement between a measured quantity value and a true quantity value of a measurand".

\[ acc(\rho) = \sup \{ | D(y_1, y_2) - d(x_1, x_2) | : (x_1, y_1) \in \rho, (x_2, y_2) \in \rho \} \]
Exploring space

Resolution: "smallest change in a quantity being measured that causes a perceptible change in the corresponding indication".

\[ res(\rho)(y) = \sup \{d(x_1, x_2) : (x_1, y) \in \rho, (x_2, y) \in \rho\} \]
Exploring space

Precision: "closeness of agreement between indications or measured quantity values obtained by replicate measurements on the same or similar objects under specified conditions".

\[ prec(\rho)(x) = \sup \{ D(y_1, y_2) : (x, y_1) \in \rho, (x, y_2) \in \rho \} \]
Exploring space

"Cartographic generalization is the method whereby information is selected and represented on a map in a way that adapts to the scale of the display medium of the map, not necessarily preserving all intricate geographical or other cartographic details”.

Let $\rho \subseteq X \times Y$ be a relation such that $\text{dom } \rho$ is $\varepsilon$-dense in $(X, d)$ and $\text{im } \rho$ is $\mu$-dense in $(Y, D)$. We define then $\bar{\rho} \subseteq X \times Y$ by: $(x, y) \in \bar{\rho}$ if there is $(x', y') \in \rho$ such that $d(x, x') \leq \varepsilon$ and $D(y, y') \leq \mu$. 
Exploring space

(a) \( \text{res}(\rho) \leq \text{acc}(\rho) \), \( \text{prec}(\rho) \leq \text{acc}(\rho) \),

(b) \( \text{res}(\rho) + 2\varepsilon \leq \text{res}(\bar{\rho}) \leq \text{acc}(\rho) + 2(\varepsilon + \mu) \),

(c) \( \text{prec}(\rho) + 2\mu \leq \text{prec}(\bar{\rho}) \leq \text{acc}(\rho) + 2(\varepsilon + \mu) \),

(d) \( |\text{acc}(\bar{\rho}) - \text{acc}(\rho)| \leq 2(\varepsilon + \mu) \).
ARE THERE a priori LOWER BOUNDS ON THE ACCURACY?
Gromov-Hausdorff distance

$\mu > 0$ is admissible for the pair of spaces $(X,d), (Y,D)$ if there is a relation $\rho \subset X \times Y$ such that

$\text{dom } \rho = X$,

$\text{im } \rho = Y$,

$\text{acc}(\rho) \leq \mu$.

The Gromov-Hausdorff distance between $(X,d)$ and $(Y,D)$ is

$$d_{GH}((X,d),(Y,D)) = \inf \{ \mu \ , \ \text{admissible} \}$$
A map of \((X, d)\) into \((Y, D)\), at scale \(\varepsilon > 0\) is a map of \((X, \frac{1}{\varepsilon}d)\) into \((Y, D)\).

In cartography, maps of the same territory done at smaller and smaller scales (smaller and smaller \(\varepsilon\)) must have the property:

- at the same accuracy and precision, the resolution has to become smaller and smaller.
(Y, D, y) (y ∈ Y) represents the (pointed unit ball in the) metric tangent space at x ∈ X of (X, d) if there exist a pair formed by:

- a "zoom sequence", that is a map
  \[(\varepsilon, x) \in (0, 1] \times X \mapsto \rho^x_{\varepsilon} \subset (\overline{B}(x, \varepsilon), \frac{1}{\varepsilon}d) \times (Y, D)\]
  such that \(\text{dom } \rho^x_{\varepsilon} = \overline{B}(x, \varepsilon)\), \(\text{im } \rho^x_{\varepsilon} = Y\), \((x, y) \in \rho^x_{\varepsilon}\) for any \(\varepsilon \in (0, 1]\) and

- a "zoom modulus" \(F : (0, 1) \rightarrow [0, +\infty)\) such that \(\lim_{\varepsilon \rightarrow 0} F(\varepsilon) = 0\),

such that for all \(\varepsilon \in (0, 1)\) we have \(\text{acc}(\rho^x_{\varepsilon}) \leq F(\varepsilon)\).
Scale

accuracy:

$$\sup \left\{ \left| D(y_1, y_2) - \frac{1}{\varepsilon} d(x_1, x_2) \right| : (x_1, y_1), (x_2, y_2) \in \rho^x_\varepsilon \right\} = O(\varepsilon)$$

precision:

$$\sup \{ D(y_1, y_2) : (u, y_1) \in \rho^x_\varepsilon, (u, y_2) \in \rho^x_\varepsilon, u \in \bar{B}(x, \varepsilon) \} = O(\varepsilon)$$

resolution:

$$\sup \{ d(x_1, x_2) : (x_1, z) \in \rho^x_\varepsilon, (x_2, z) \in \rho^x_\varepsilon, z \in Y \} = \varepsilon O(\varepsilon)$$
Let \( \epsilon, \mu \in (0, 1) \) and \( \rho^x_\epsilon \subset \overline{B}(x, \epsilon) \times \overline{B}(y, 1) \), \( \rho^x_{\epsilon \mu} \subset \overline{B}(x, \epsilon \mu) \times \overline{B}(y, 1) \)
CASCADING OF ERRORS: \[ \text{acc}(\rho_{\frac{x}{\varepsilon}, \mu}) \leq \frac{1}{\mu}O(\varepsilon) + O(\varepsilon \mu) \]
scale stable if $D^\text{Hausdorff}_\mu \left( \rho_{\varepsilon,\mu}^x, \bar{\rho}_\mu^x \right) \leq F_\mu(\varepsilon)$
Scale stability

If there is a scale stable zoom sequence $\rho_{\varepsilon}^x$ then the space $(Y, D)$ is self-similar in a neighbourhood of point $y \in Y$:

for any $(u', u''), (v', v'') \in \overline{\rho}_\mu^x$ we have:

$$D(u'', v'') = \frac{1}{\mu} D(u', v')$$

In particular $\overline{\rho}_\mu^x$ is the graph of a function.
We have a zoom sequence, a scale $\varepsilon \in (0, 1)$ and two points: $x \in X$ and $u' \in \overline{B}(y, 1)$. 

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difference at scale $\varepsilon$, from $x$ to $x_1$, as seen from $u'$

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viewpoint stable if $D_{\mu}^{Hausdorff} \left( \Delta^x(u', \cdot), \Delta^x(u', \cdot) \right) \leq F_{diff}(\varepsilon)$
$\Delta^x(u', \cdot)$ is the graph of an isometry of $(Y, D)$. 

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Foveal maps

A scale stable zoom sequence of maps can be improved such that all maps from the new zoom sequence have better accuracy near the "center" of the map $x \in X$, which justifies the name "foveal maps".

The accuracy of the restriction of each improved map

$$\phi_x^x \cap (\bar{B}(x, \varepsilon \mu) \times \bar{B}(y, \mu))$$

is bounded by $\mu F(\varepsilon \mu)$, therefore the right hand side term in the cascading of errors inequality can be improved to $2F(\varepsilon \mu)$.
ARE THERE a priori OBSTRUCTIONS FOR HAVING
- SCALE STABLE - VIEWPOINT STABLE - FOVEAL
ZOOM MAPS FROM \((X, d)\) INTO \((Y, D)\)?
Dilation structures

Dilation structure (a generalization): a foveal, scale stable, viewpoint stable sequence of zoom maps of \((X, d)\) into \((Y, D)\).

Suppose there is a dilation structure of \((X, d)\) into \((Y, D)\). Then for any \(x \in X\) the space \((Y, D)\) admits a local group operation \((v, w) \mapsto v \cdot x w\) such that:

- all left translations are \(D\) isometries

- the difference relation \(\Delta^x(u, \cdot)\) is the graph of the left translation \(v \mapsto u^{-1} \cdot x v\)

Moreover, the local group operation admits a one-parameter family of isomorphisms, which have as graphs the dilation relations \(\bar{\rho}_\mu^x\).
Conical groups

A CONICAL GROUP is a pair \((N, \delta)\) such that:

- \(N\) is a topological group,

- \(\delta\) is an action of a commutative group (say \((0, +\infty)\)) by automorphisms on \(N\), such that

\[
\lim_{\varepsilon \to 0} \delta_\varepsilon x = e
\]

uniformly with respect to \(x\) in a compact neighbourhood of the identity \(e\).

NORMED CONICAL GROUP:

- there is also a group norm \(\| \cdot \| : N \to [0, +\infty)\), \(\|xy\| \leq \|x\| + \|y\|\) ...

- such that \(\|\delta_\varepsilon x\| = \varepsilon \|x\|\).
Conical groups

(Siebert) Locally compact, connected conical groups are Carnot groups.

(Goldbring) same statement for local groups.

Examples:

- \((\mathbb{R}^n, +)\) with \(\delta_\varepsilon x = \varepsilon x\), and a usual norm

- Heisenberg group: \(H(n) = \mathbb{R}^{2n} \times \mathbb{R}\) with \((X, x) \cdot (Y, y) = (X + Y, x + y + \frac{1}{2}\omega(X, Y))\) and \(\delta_\varepsilon(X, x) = (\varepsilon X, \varepsilon^2 x)\). Norm given by a Carnot-Carathéodory left invariant distance.

- there are also plenty of examples of non connected locally compact conical groups (coming from ultrametric spaces).
Conical groups

Conical groups appear in (not exclusive):

- Gromov polynomial growth theorem: a finitely generated group with polynomial growth (i.e. number of elements which can be expressed as a product of at most \( n \) generators grows like a polynomial in \( n \)) is virtually (up to factorization by a finite group) conical.

- Mitchell theorem: the metric tangent space at a point in a regular sub-riemannian manifold is a conical group.

- Pansu-Rademacher theorem: a Lipschitz function between two Carnot groups is derivable (see later) almost everywhere.

- Tao-Green-Breuillard theorem: an approximate group is roughly equivalent with a ball in a normed conical group.
Differentiability

Take two dilation structures:

- of \((X_1, d_1)\) into \((Y_1, D_1)\)

- of \((X_2, d_2)\) into \((Y_2, D_2)\)

and a function \(f : X_1 \rightarrow X_2\). For any \(x \in X_1\) and \(\varepsilon > 0\) consider the relation

\[ \rho_{\varepsilon}^f(x) f(\rho_{\varepsilon}^x)^{-1} \]

from \(Y_1\) into \(Y_2\). If this relation converges (w.r.t. Hausdorff distance) \textbf{TO THE GRAPH OF A MORPHISM OF CONICAL GROUPS} then we say \(f\) is differentiable in \(x\).
Non-euclidean analysis

A dilation structure of \((X_1, d_1)\) into \((Y_1, D_1)\)

looks down at

another dilation structure of \((X_2, d_2)\) into \((Y_2, D_2)\)

if for any \(x \in X_1\) there is a neighbourhood of \(x\) and a bijective map, from it to a neighbourhood of \(f(x)\), which is differentiable everywhere, uniformly w.r.t. \(x \in X\).

Two dilation structures are equivalent if each looks down at the other.

An equivalence class of dilation structures is called an ”analysis”. 
Non-euclidean analysis

Examples:

- a real manifold endowed with the dilation structure given by an atlas is equivalent with $\mathbb{R}^n$.

- a metric contact manifold has a dilation structure equivalent with the one of a Heisenberg group.

- if two Carnot groups have equivalent left invariant dilation structures (over themselves) then they are isomorphic as groups.

- any sub-riemannian (or Carnot-Carathéodory) manifold has a dilation structure which looks down at any riemannian structure over the same manifold.
Non-euclidean analysis

Do we really need distances?

Is this a metric phenomenon?

NO.

Google search:

"metric spaces with dilations"

"dilation structures"

"emergent algebras"