Non-euclidean analysis of dilation structures

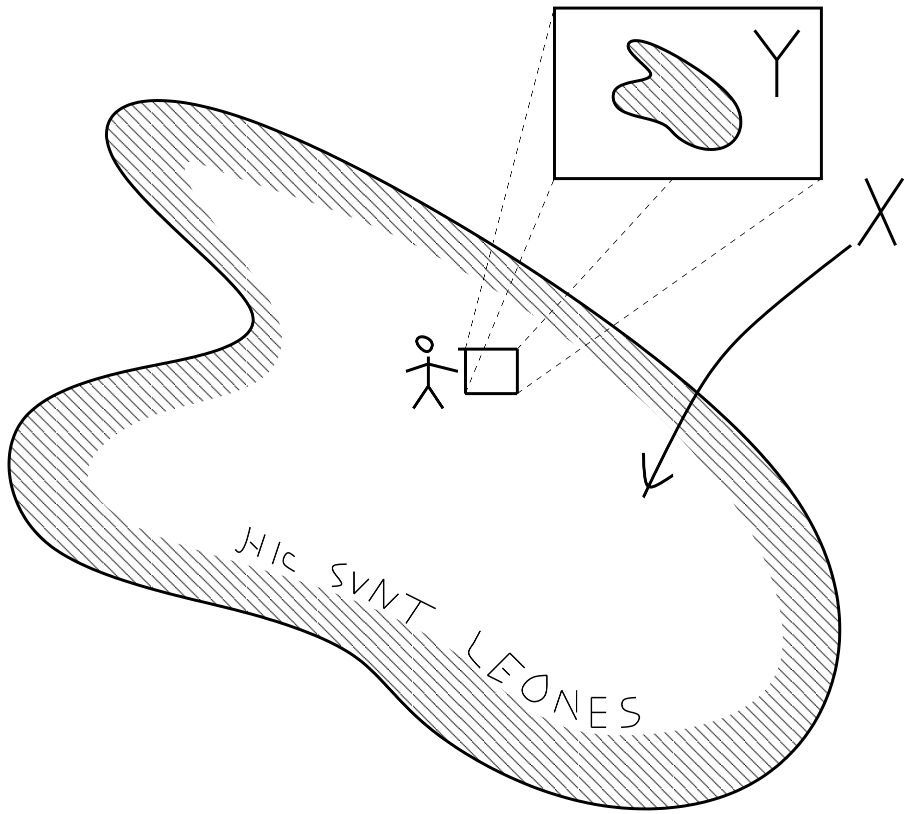
M. Buliga

IMAR

Institute of Mathematics
of the Romanian Academy

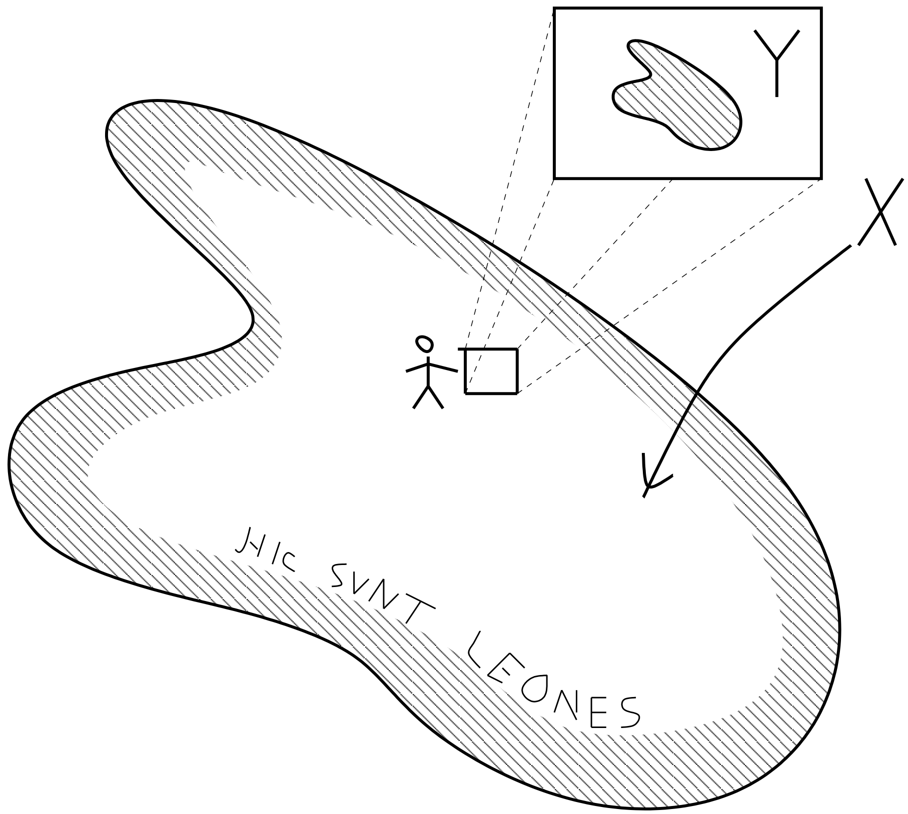
marius.buliga@imar.ro

Exploring space



Imagine that the metric space (X, d) represents a territory. We want to make maps of (X, d) in the metric space (Y, D) (a piece of paper, or a scaled model).

Exploring space

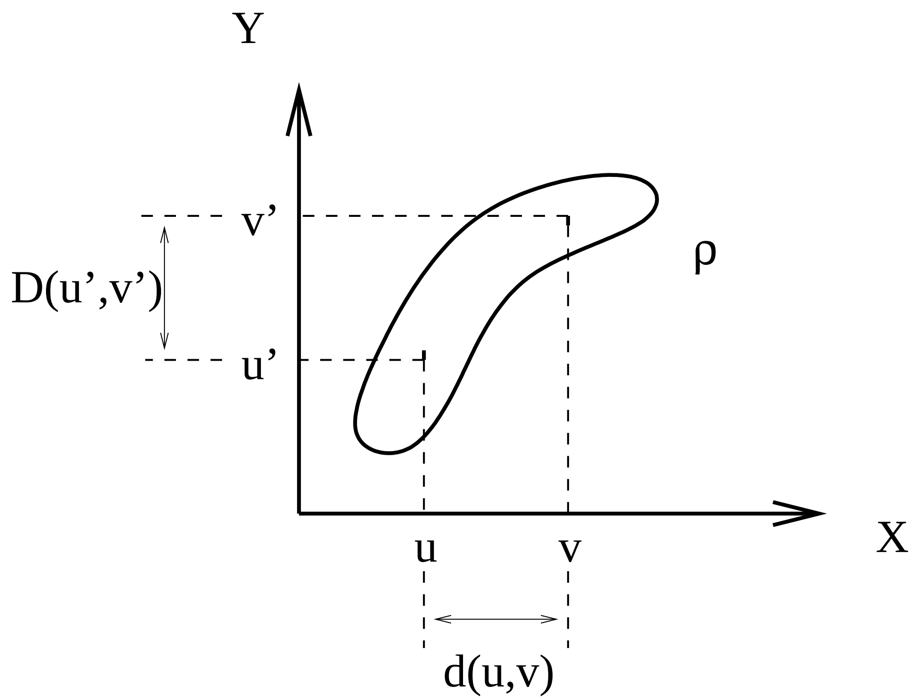


We need many maps, at several scales $\varepsilon_1 > \varepsilon_2 > ... > \varepsilon_n$.

An atlas of compatible maps, a manifold?

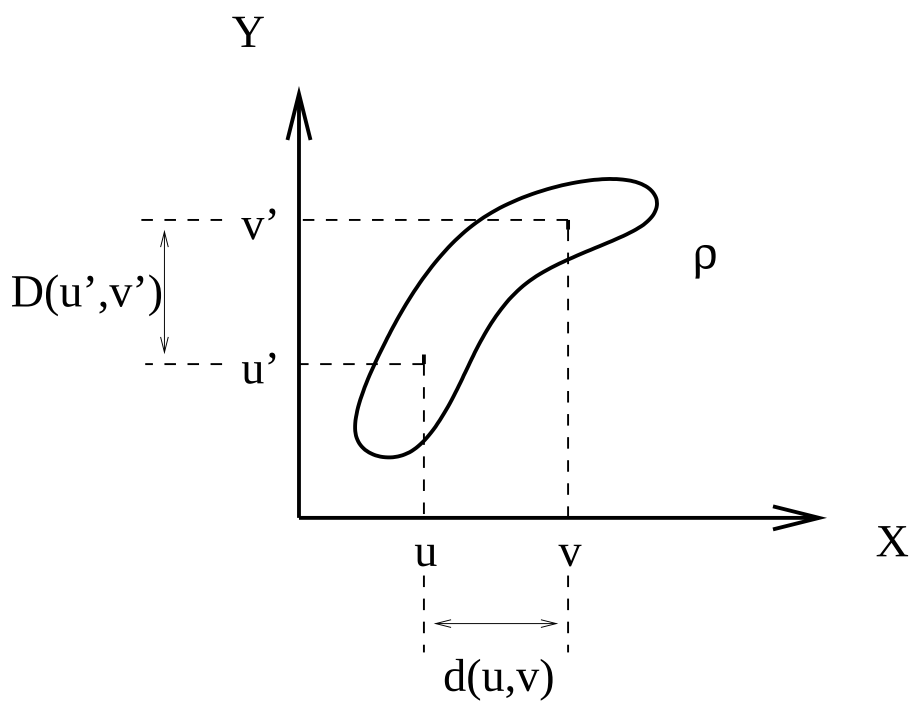
Exploring space

A model of a map of (X,d) in (Y,D) is a relation $\rho \subset X \times Y$.



How good is the map? Look at: $\mid d(u,v) - D(u',v') \mid$.

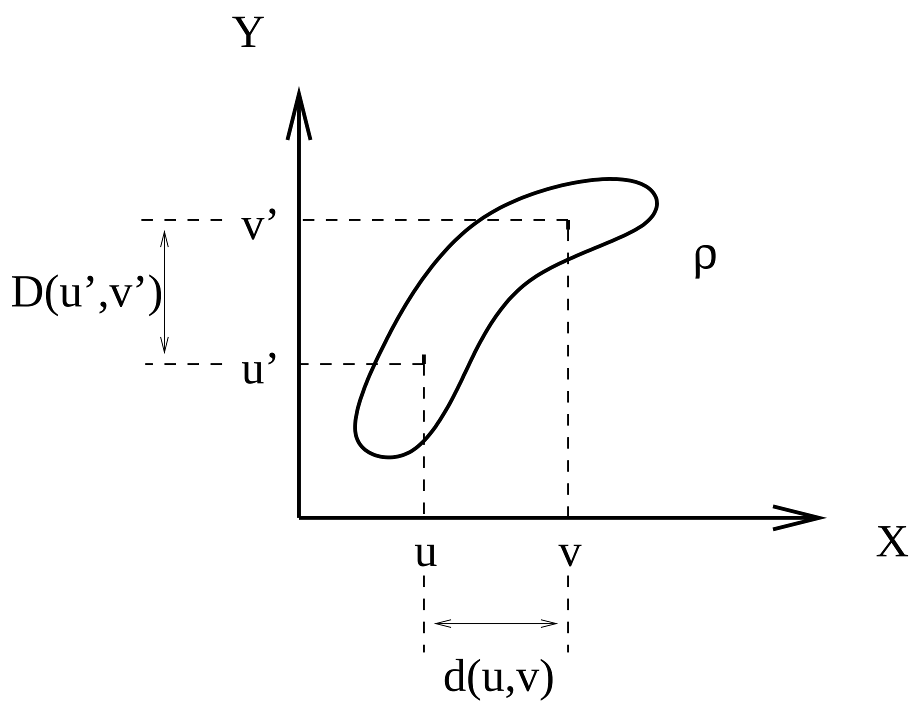
Exploring space



Accuracy: "closeness of agreement between a measured quantity value and a true quantity value of a measurand".

$$acc(\rho) = \sup \{ \mid D(y_1, y_2) - d(x_1, x_2) \mid \ : \ (x_1, y_1) \in \rho \, , \, (x_2, y_2) \in \rho \}$$

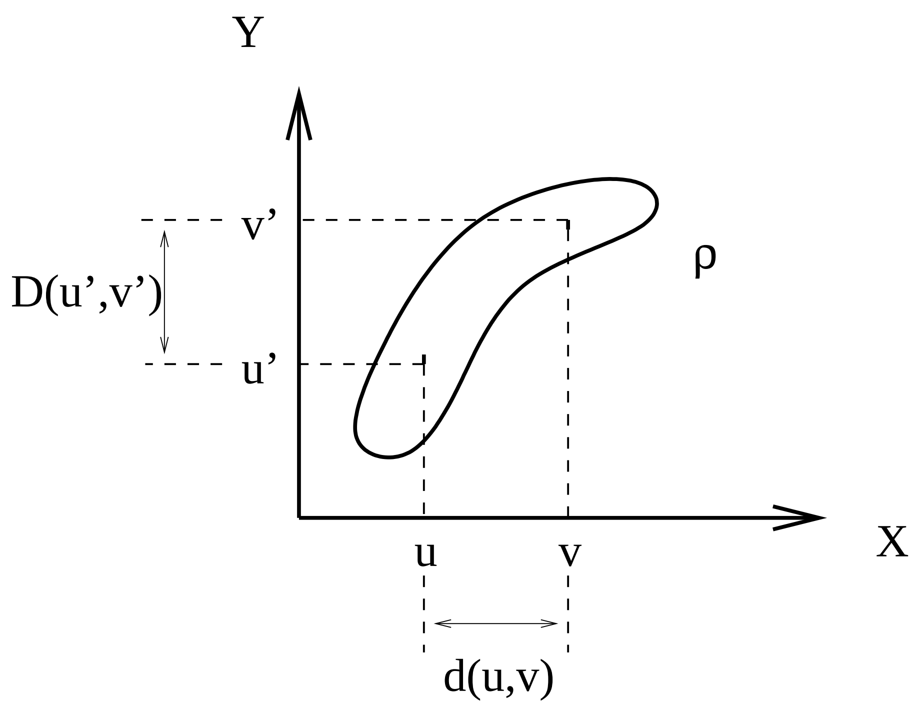
Exploring space



Resolution: ”smallest change in a quantity being measured that causes a perceptible change in the corresponding indication”.

$$res(\rho)(y) = \sup \{ d(x_1, x_2) \, : \, (x_1, y) \in \rho \, , \, (x_2, y) \in \rho \}$$

Exploring space



Precision: "closeness of agreement between indications or measured quantity values obtained by replicate measurements on the same or similar objects under specified conditions".

$$prec(\rho)(x) = \sup \{D(y_1, y_2) \, : \, (x, y_1) \in \rho \, , \, (x, y_2) \in \rho\}$$

Exploring space

"Cartographic generalization is the method whereby information is selected and represented on a map in a way that adapts to the scale of the display medium of the map, not necessarily preserving all intricate geographical or other cartographic details".

Let $\rho \subset X \times Y$ be a relation such that $dom\ \rho$ is ε-dense in (X,d) and $im\ \rho$ is μ-dense in (Y,D). We define then $\bar{\rho} \subset X \times Y$ by: $(x,y) \in \bar{\rho}$ if there is $(x',y') \in \rho$ such that $d(x,x') \leq \varepsilon$ and $D(y,y') \leq \mu$.

Exploring space

(a) $res(\rho) \leq acc(\rho)$, $prec(\rho) \leq acc(\rho)$,

(b) $res(\rho) + 2\varepsilon \leq res(\bar{\rho}) \leq acc(\rho) + 2(\varepsilon + \mu)$,

(c) $prec(\rho) + 2\mu \leq prec(\bar{\rho}) \leq acc(\rho) + 2(\varepsilon + \mu)$,

(d) $\mid acc(\bar{\rho}) - acc(\rho) \mid \leq 2(\varepsilon + \mu)$.

ARE THERE *a priori* LOWER BOUNDS ON THE ACCURACY?

Gromov-Hausdorff distance

$\mu > 0$ is adimissible for the pair of spaces (X, d), (Y, D) if there is a relation $\rho \subset X \times Y$ such that

$dom \ \rho = X$,

$im \ \rho = Y$,

$acc(\rho) \leq \mu$.

The Gromov-Hausdorff distance between (X, d) and (Y, D) is

$$d_{GH}((X, d), (Y, D)) = \text{inf} \, \{\mu \ , \ \ \text{admissible} \ \}$$

Scale

A map of (X, d) into (Y, D), at scale $\varepsilon > 0$ is a map of $(X, \frac{1}{\varepsilon} d)$ into (Y, D).

In cartography, maps of the same territory done at smaller and smaller scales (smaller and smaller ε) must have the property:

- at the same accuracy and precision, the resolution has to become smaller and smaller.

Scale

(Y, D, y) $(y \in Y)$ represents the (pointed unit ball in the) metric tangent space at $x \in X$ of (X, d) if there exist a pair formed by:

- a "zoom sequence", that is a map

$$(\varepsilon, x) \in (0,1] \times X \mapsto \rho_\varepsilon^x \subset (\bar{B}(x, \varepsilon), \frac{1}{\varepsilon} d) \times (Y, D)$$

such that $dom\ \rho_\varepsilon^x = \bar{B}(x, \varepsilon)$, $im\ \rho_\varepsilon^x = Y$, $(x, y) \in \rho_\varepsilon^x$ for any $\varepsilon \in (0,1]$ and

- a "zoom modulus" $F : (0,1) \to [0, +\infty)$ such that $\lim_{\varepsilon \to 0} F(\varepsilon) = 0$,

such that for all $\varepsilon \in (0,1)$ we have $acc(\rho_\varepsilon^x) \leq F(\varepsilon)$.

Scale

accuracy:

$$\sup \left\{ \mid D(y_1, y_2) - \frac{1}{\varepsilon} d(x_1, x_2) \mid \ : \ (x_1, y_1), \, (x_2, y_2) \in \rho_\varepsilon^x \right\} = \mathcal{O}(\varepsilon)$$

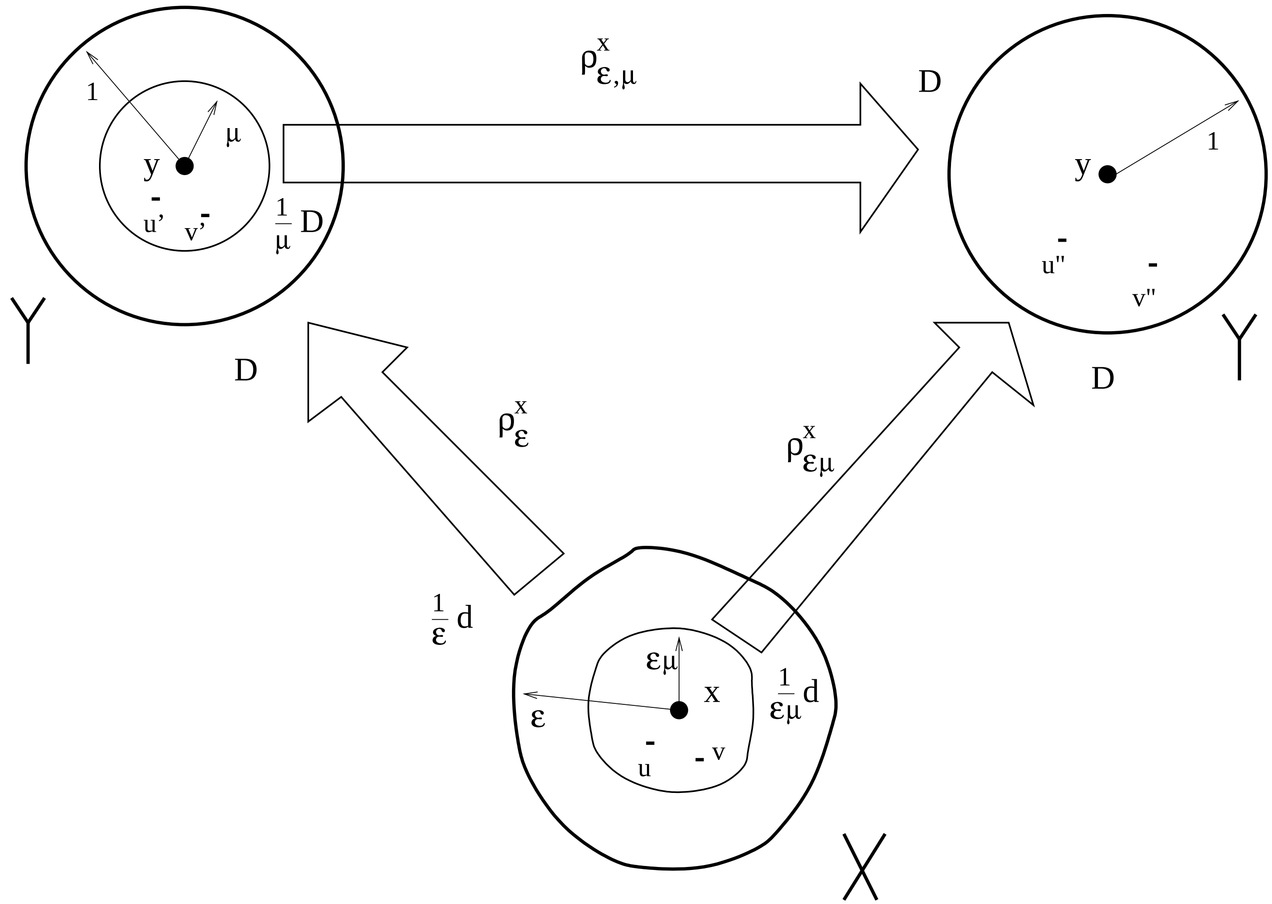
precision:

$$\sup \left\{ D(y_1, y_2) \ : \ (u, y_1) \in \rho_\varepsilon^x \, , \, (u, y_2) \in \rho_\varepsilon^x \, , \, u \in \bar{B}(x, \varepsilon) \right\} = \mathcal{O}(\varepsilon)$$

resolution:

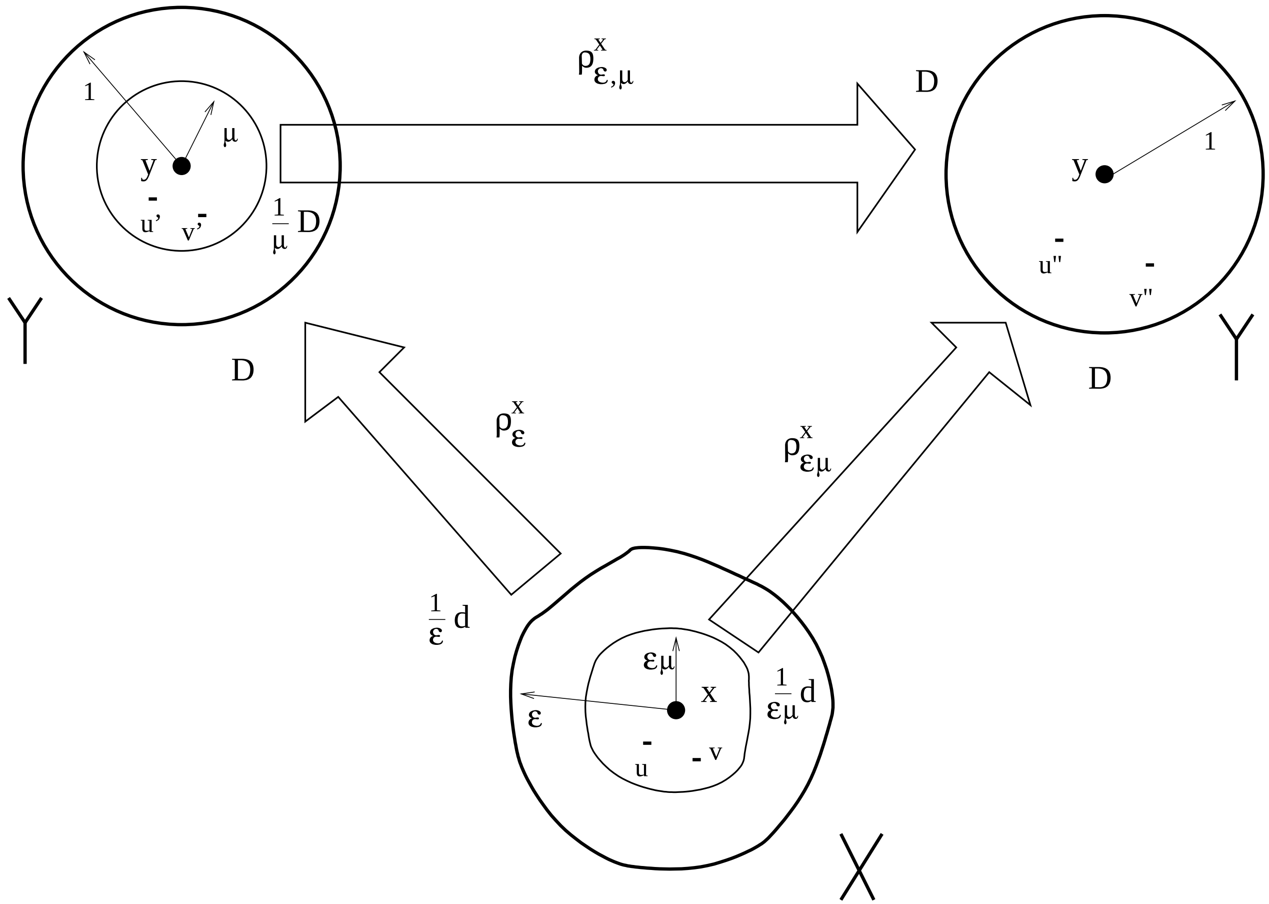
$$\sup \left\{ d(x_1, x_2) \ : \ (x_1, z) \in \rho_\varepsilon^x \, , \, (x_2, z) \in \rho_\varepsilon^x \, , \, z \in Y \right\} = \varepsilon \, \mathcal{O}(\varepsilon)$$

Scale stability



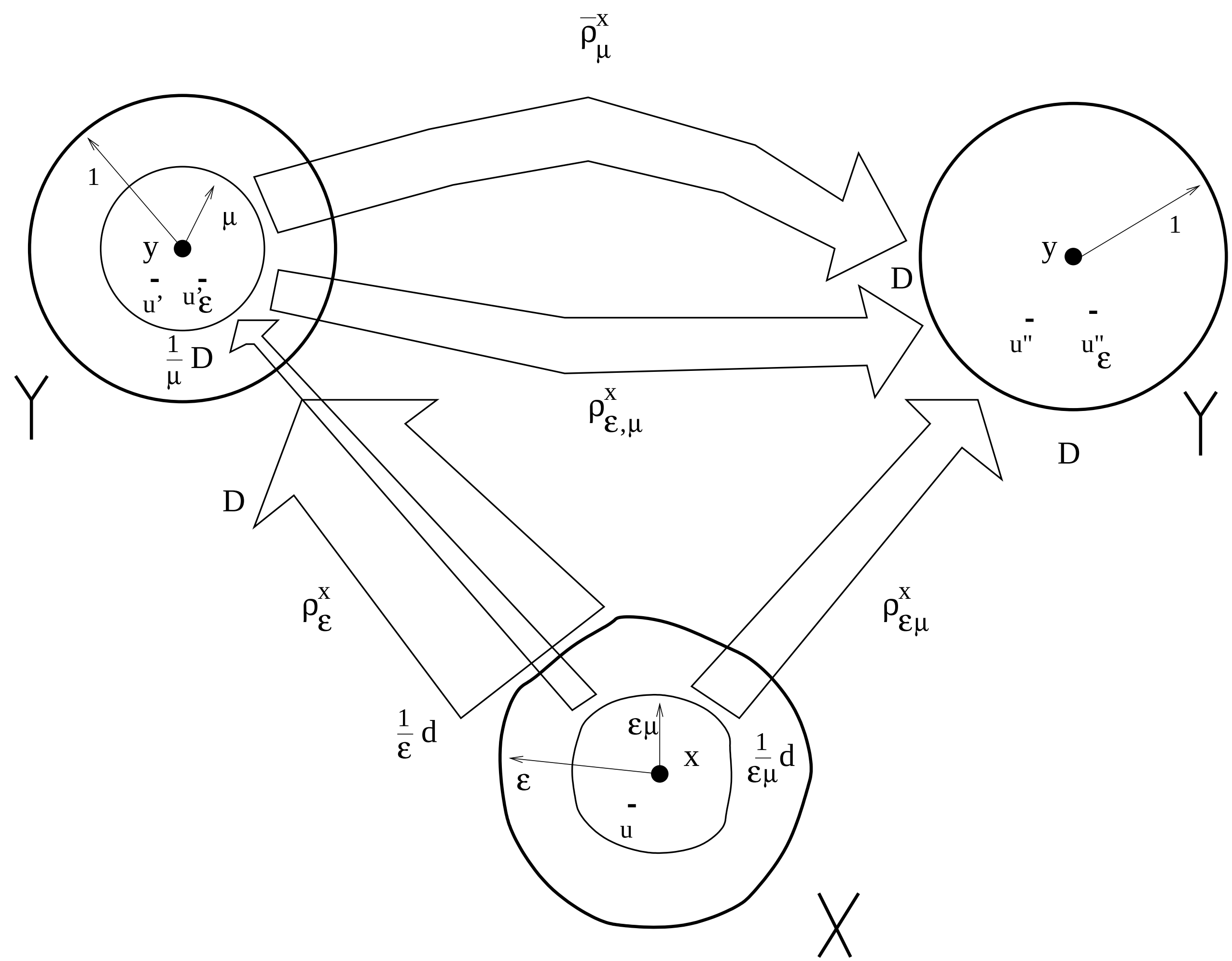
Let $\varepsilon, \mu \in (0,1)$ and $\rho^x_\varepsilon \subset \bar{B}(x,\varepsilon) \times \bar{B}(y,1)$, $\rho^x_{\varepsilon\mu} \subset \bar{B}(x,\varepsilon\mu) \times \bar{B}(y,1)$

Scale stability



CASCADING OF ERRORS: $acc(\rho^x_{\varepsilon,\mu}) \leq \frac{1}{\mu}\mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\mu)$

Scale stability



scale stable if $D_\mu^{Hausdorff}\left(\rho^x_{\varepsilon,\mu}, \bar{\rho}^x_\mu\right) \leq F_\mu(\varepsilon)$

Scale stability

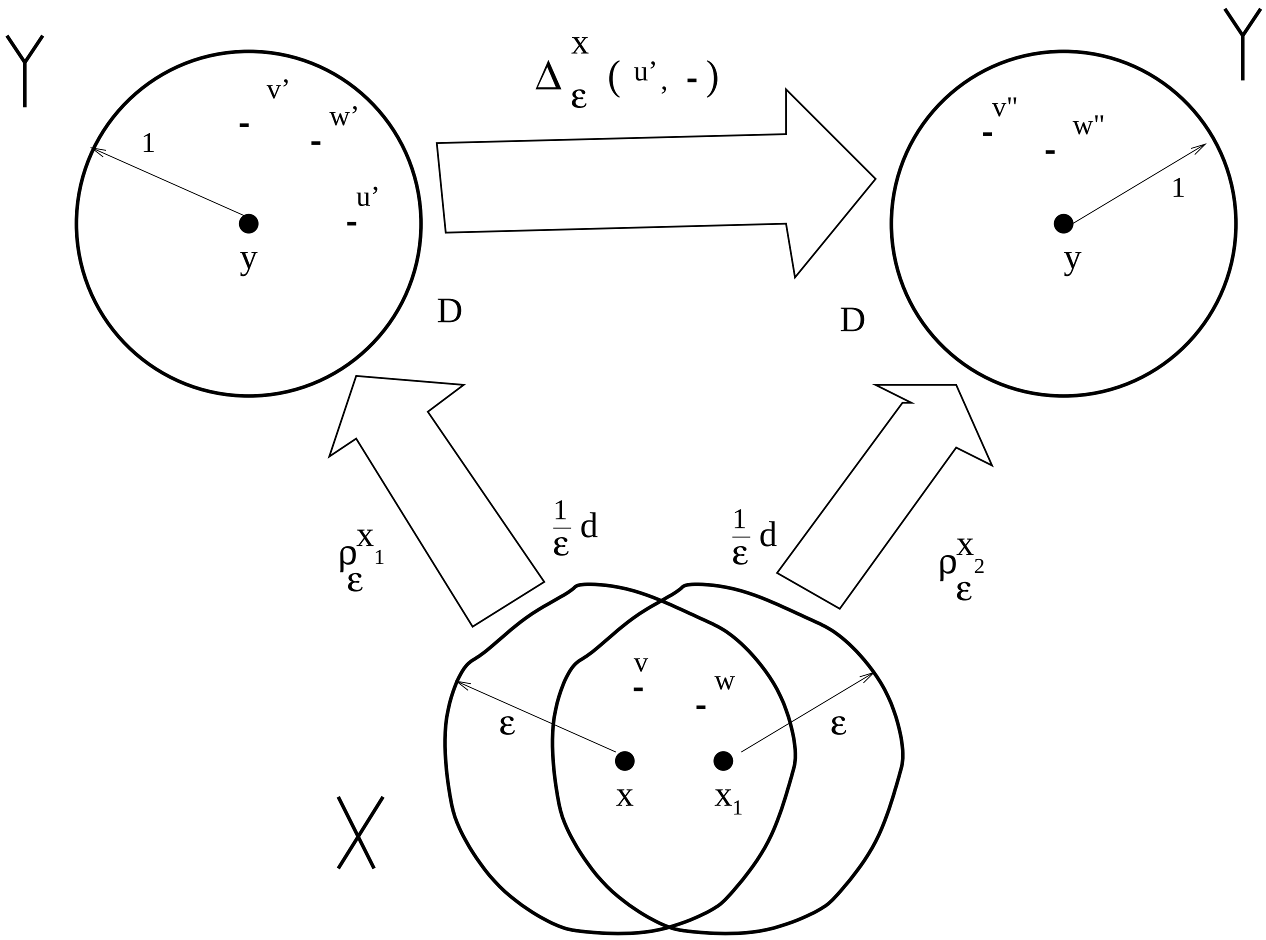
If there is a scale stable zoom sequence ρ^x_ε then the space (Y, D) is self-similar in a neighbourhood of point $y \in Y$:

for any $(u', u''), (v', v'') \in \bar{\rho}^x_\mu$ we have:

$$D(u'', v'') = \frac{1}{\mu} D(u', v')$$

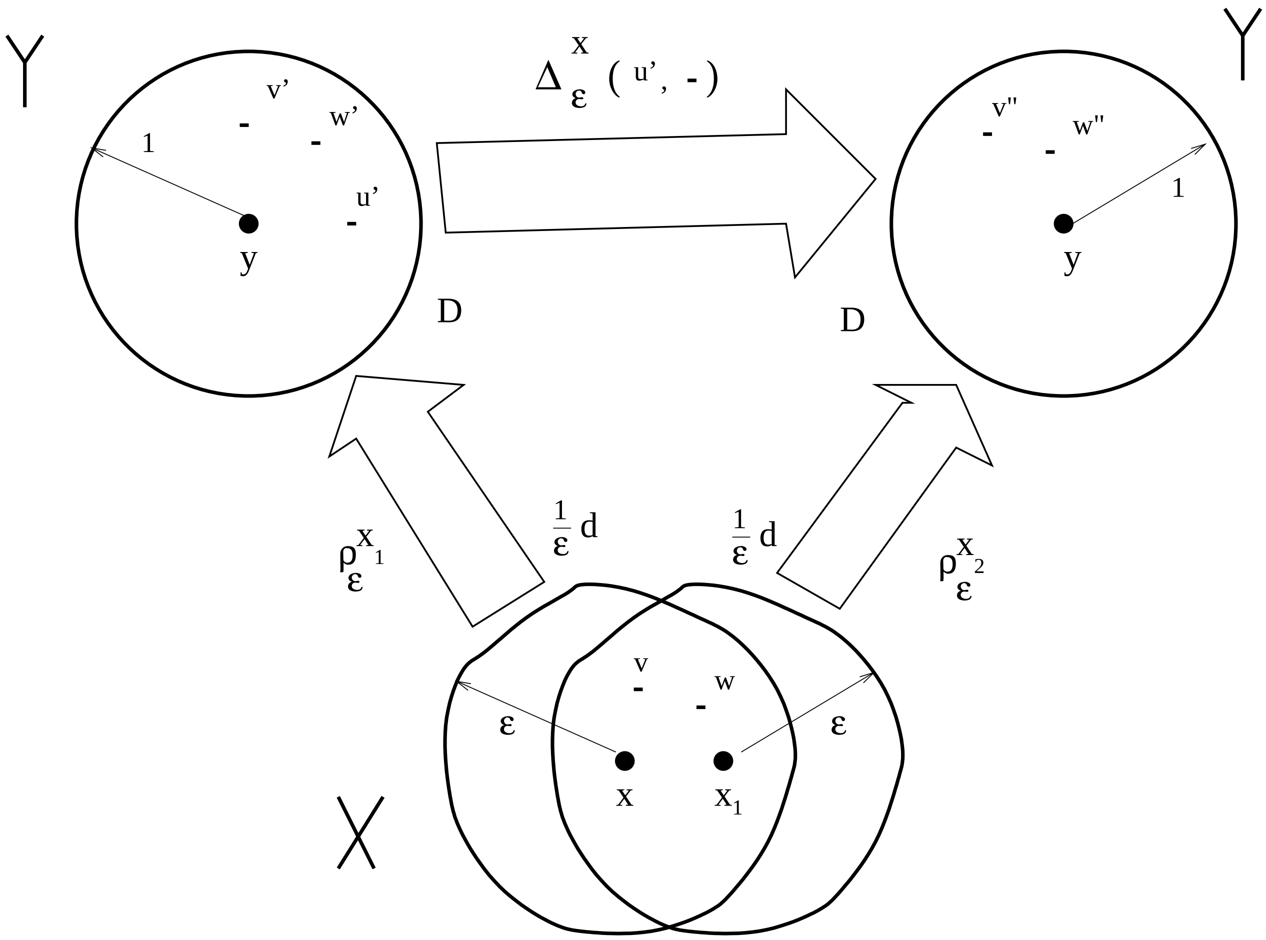
In particular $\bar{\rho}^x_\mu$ is the graph of a function.

Viewpoint stability



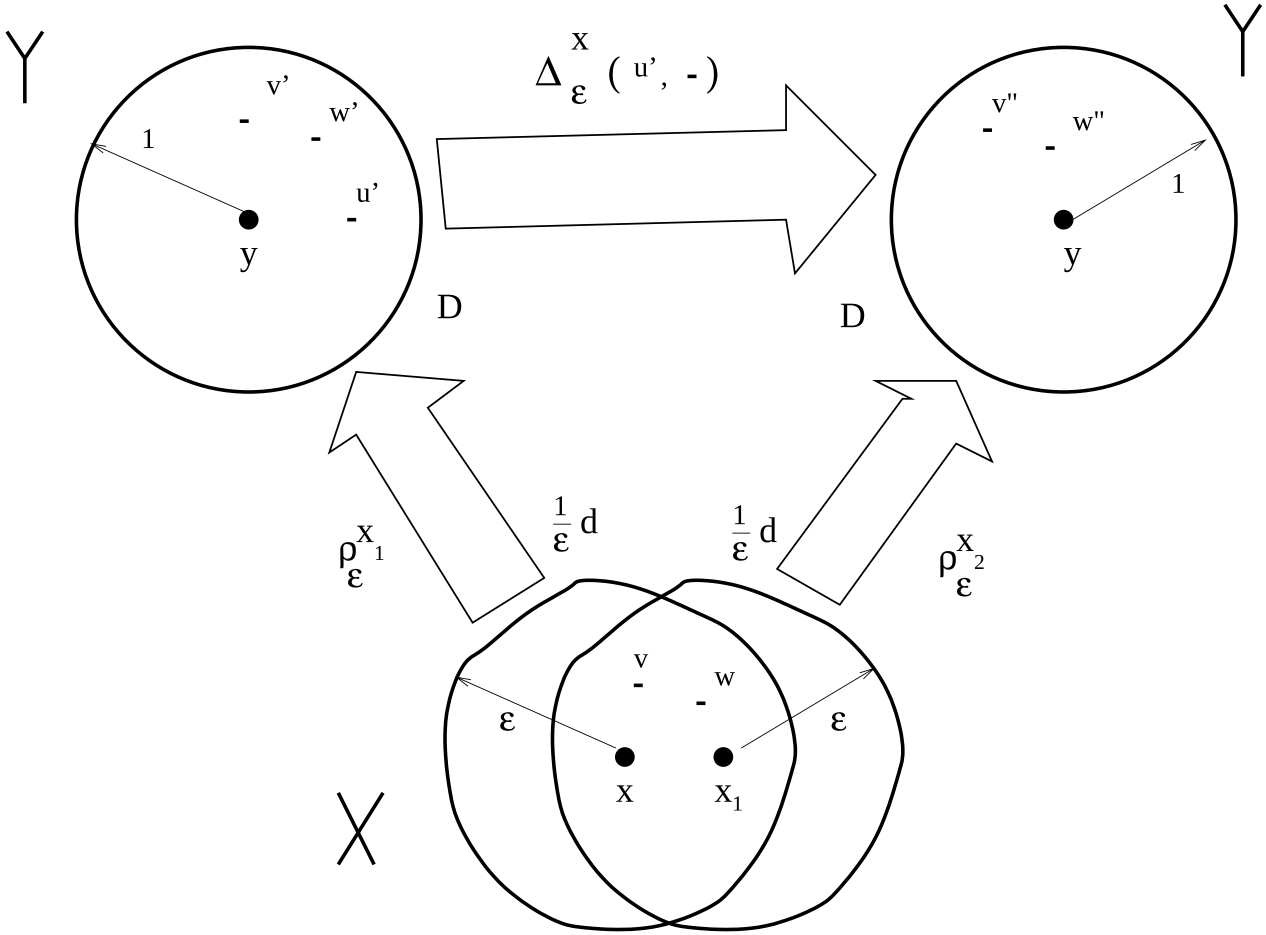
We have a zoom sequence, a scale $\varepsilon \in (0, 1)$ and two points: $x \in X$ and $u' \in \bar{B}(y, 1)$.

Viewpoint stability



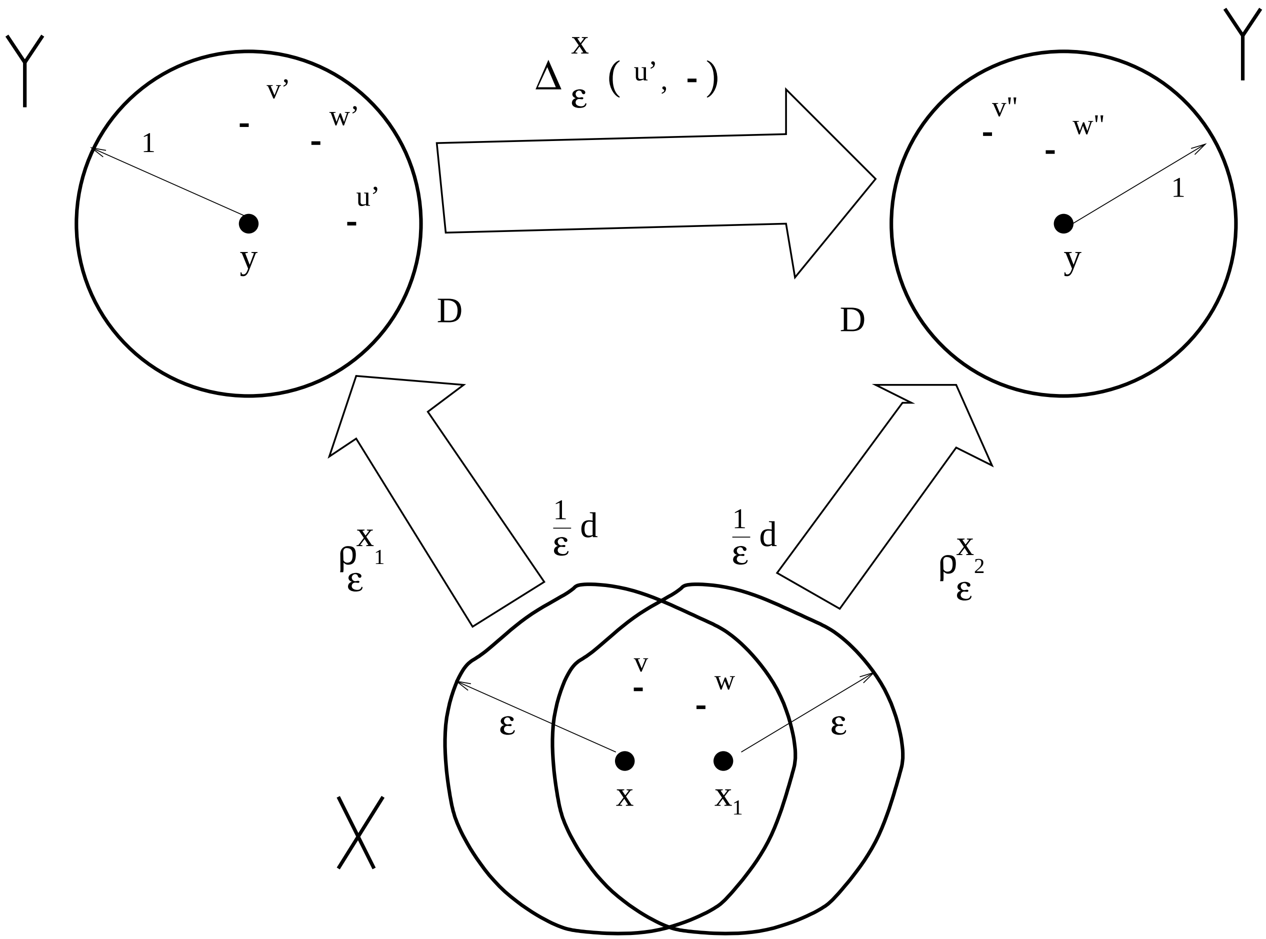
difference at scale ε, from x to x_1, as seen from u'

Viewpoint stability



viewpoint stable if $D_{\mu}^{Hausdorff}\left(\Delta_{\varepsilon}^{x}(u',\cdot), \Delta^{x}(u',\cdot)\right) \leq F_{diff}(\varepsilon)$

Viewpoint stability



$\Delta^x(u', \cdot)$ is the graph of an isometry of (Y, D).

Foveal maps

A scale stable zoom sequence of maps can be improved such that all maps from the new zoom sequence have better accuracy near the "center" of the map $x \in X$, which justifies the name "foveal maps".

The accuracy of the restriction of each improved map

$$\phi_{\varepsilon}^{x} \cap (\bar{B}(x, \varepsilon\mu) \times \bar{B}(y, \mu))$$

is bounded by $\mu F(\varepsilon\mu)$, therefore the right hand side term in the cascading of errors inequality can be improved to $2F(\varepsilon\mu)$.

ARE THERE *a priori* OBSTRUCTIONS FOR HAVING

- SCALE STABLE - VIEWPOINT STABLE - FOVEAL

ZOOM MAPS FROM (X, d) INTO (Y, D)?

Dilation structures

Dilation structure (a generalization): a foveal, scale stable, viewpoint stable sequence of zoom maps of (X, d) into (Y, D).

Suppose there is a dilation structure of (X, d) into (Y, D). Then for any $x \in X$ the space (Y, D) admits a local group operation $(v, w) \mapsto v \cdot_x w$ such that:

- all left translations are D isometries

- the difference relation $\Delta^x(u, \cdot)$ is the graph of the left translation $v \mapsto u^{-1} \cdot_x v$

Moreover, the local group operation admits a one-parameter family of isomorphisms, which have as graphs the dilation relations $\bar{\rho}^x_\mu$.

Conical groups

A CONICAL GROUP is a pair (N, δ) such that:

- N is a topological group,

- δ is an action of a commutative group (say $(0, +\infty)$) by automorphisms on N, such that

$$\lim_{\varepsilon \to 0} \delta_\varepsilon x = e$$

uniformly with respect to x in a compact neighbourhood of the identity e.

NORMED CONICAL GROUP:

- there is also a group norm $\| \cdot \| : N \to [0, +\infty)$, $\|xy\| \leq \|x\| + \|y\|$ ...

- such that $\|\delta_\varepsilon x\| = \varepsilon \|x\|$.

Conical groups

(Siebert) Locally compact, connected conical groups are Carnot groups.

(Goldbring) same statement for local groups.

Examples:

- $(\mathbb{R}^n, +)$ with $\delta_\varepsilon x = \varepsilon x$, and a usual norm

- Heisenberg group: $H(n) = \mathbb{R}^{2n} \times \mathbb{R}$ with $(X, x) \cdot (Y, y) = (X + Y, x + y + \frac{1}{2}\omega(X, Y)$ and $\delta_\varepsilon(X, x) = (\varepsilon X, \varepsilon^2 x)$. Norm given by a Carnot-Carathéodory left invariant distance.

- there are also plenty of examples of non connected locally compact conical groups (coming from ultrametric spaces).

Conical groups

Conical groups appear in (not exclusive):

- Gromov polynomial growth theorem: a finitely generated group with polynomial growth (i.e. number of elements which can be expressed as a product of at most n generators grows like a polynomial in n) is virtually (up to factorization by a finite group) conical.

- Mitchell theorem: the metric tangent space at a point in a regular sub-riemannian manifold is a conical group.

- Pansu-Rademacher theorem: a Lipschitz function between two Carnot groups is derivable (see later) almost everywhere.

- Tao-Green-Breuillard theorem: an approximate group is roughly equivalent with a ball in a normed conical group.

Differentiability

Take two dilation structures:

- of (X_1, d_1) into (Y_1, D_1)

- of (X_2, d_2) into (Y_2, D_2)

and a function $f : X_1 \rightarrow X_2$. For any $x \in X_1$ and $\varepsilon > 0$ consider the relation

$$\rho_\varepsilon^{f(x)} f \left(\rho_\varepsilon^x\right)^{-1}$$

from Y_1 into Y_2. If this relation converges (w.r.t. Hausdorff distance) TO THE GRAPH OF A MORPHISM OF CONICAL GROUPS then we say f is differentiable in x.

Non-euclidean analysis

A dilation structure of (X_1, d_1) into (Y_1, D_1)

looks down at

another dilation structure of (X_2, d_2) into (Y_2, D_2)

if for any $x \in X_1$ there is a neighbourhood of x and a bijective map, from it to a neighbourhood of $f(x)$, which is differentiable everywhere, uniformly w.r.t. $x \in X$.

Two dilation structures are equivalent if each looks down at the other.

An equivalence class of dilation structures is called an "analysis".

Non-euclidean analysis

Examples:

- a real manifold endowed with the dilation structure given by an atlas is equivalent with $\mathbb{R}^n$.

- a metric contact manifold has a dilation structure equivalent with the one of a Heisenberg group.

- if two Carnot groups have equivalent left invariant dilation structures (over themselves) then they are isomorphic as groups.

- any sub-riemannian (or Carnot-Carathéodory) manifold has a dilation structure which looks down at any riemannian structure over the same manifold.

Non-euclidean analysis

Do we really need distances?

Is this a metric phenomenon?

NO.

Google search:

"metric spaces with dilations"

"dilation structures"

"emergent algebras"